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THE PROBLEM OF CONSTRUCTING A LYAPUNOV FUNCTION *

A.P. BLINOV

An algorithm which, for a wide class of problems, enables a Lyapunov function with a negative-sign derivative to be reconstructed as a Lyapunov function with a negative-definite derivative, is proposed. This algorithm supplements the well-known method /1/ of reconstructing a Lyapunov function. Examples are considered.

Consider a set of differential equations of perturbed motion

$$x_i^{\cdot} = f_i(x),\ f(0) = 0,\ x \in R^n,\ f_i(x) \in C^1(\Omega),\ \{0\} \in \Omega \subset R^n \tag{1}$$

We will assume that for (1) Lyapunov's function $V_0(x)$, which is positive definite in the domain Ω and whose time-derivative is non-positive in this domain and vanishes in the manifold $M \subset \Omega$ by virtue of Eqs.(1), is known.

We shall formulate the problem of determining the functions $V_\nu(x)$ $(\nu \leqslant n-1)$ and the constants $\mu_\nu > 0$, for which the sum

$$V(x) = V_0(x) + \sum_{\nu=1}^{p} \mu_\nu V_\nu(x), \quad p \leqslant n-1 \tag{2}$$

(the quantity p is refined while solving the problem) will be positive definite, and its time derivative is, by virtue of (1), a negative-definite function in Ω.

We shall show that for the additional assumptions introduced below this problem has the following solution.

Suppose the manifold M is described by the equations $S_1(x) = 0, \ldots, S_m(x) = 0$, which are solvable in Ω with respect to certain m variables, for example

$$x_j = x_j^\circ(x_{m+1}, \ldots, x_n),\ x_j^\circ(0) = 0,\ j = 1, \ldots, m$$

We shall determine the functions f_i° and Φ_k $(i, k = 1, \ldots, n)$ using the equations

$$f_i^\circ(x_{m+1}, \ldots, x_n) = f_i(x_1^\circ(x_{m+1}, \ldots, x_n), \ldots, x_m^\circ(x_{m+1}, \ldots, x_n), x_{m+1}, \ldots, x_n) \tag{3}$$

$$\Phi_k(x_k, x_{m+1}, \ldots, x_n) = -\int_0^{x_k} f_k^\circ(x_{m+1}, \ldots, x_n)\, dx_k + \Phi_{0k} \tag{4}$$

Here Φ_{0k} is an arbitrary function of the coordinates $x_{m+1}, \ldots, x_n$, in a number of which the coordinate x_k does not occur, and $\Phi_{0k}(0) = 0$. (When $k \geqslant m+1$ x_k is omitted in the left-hand side of (4)).

If the functions f_i do not depend on $x_{m+1}, \ldots, x_n$, we will assume that $f_i^\circ \equiv 0$.

We shall determine the function $V_{*1}(x)$ in the form of the sum

$$V_{*1}(x) = \sum_{k=1}^{n} \lambda_k \Phi_k(x)$$

in which the constants λ_k will be determined below.

We shall write the time-derivative of this function by virtue of (1)

$$V_{*1}^{\cdot}(x) = \sum_{i=1}^{n} f_i(x) \sum_{k=1}^{n} \lambda_k \frac{\partial \Phi_k(x)}{\partial x_i}$$

Since for $i \leqslant m$

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$$\sum_{k=1}^{n} \lambda_k \frac{\partial \Phi_k(x)}{\partial x_i} = -\lambda_i f_i^\circ(x)$$

after regrouping the terms we will obtain

$$V_{*1}^{\cdot}(x) = -\sum_{i=1}^{m} \lambda_i f_i(x) f_i^\circ(x) + \sum_{k=m+1}^{n} f_k(x) \sum_{i=1}^{m} \lambda_i \frac{\partial \Phi_i(x)}{\partial x_k} + \sum_{i,\, k=m+1}^{n} \lambda_i f_i(x) \frac{\partial \Phi_k}{\partial x_i} \tag{5}$$

In Eq.(5) the first sum is non-positive on M and the non-positive terms $f_k(x)\,\partial\Phi_k(x)/\partial x_k$ (on M), occurring in the latter sum for any $\lambda_j > 0$ $(j = 1, \ldots, n)$. Therefore, by choosing the coefficients λ_j we can attempt to satisfy the inequalities

$$V_{*1}^{\cdot}(x) \leqslant 0, \; V_{*1}^{\cdot}(x) \not\equiv 0 \tag{6}$$

if not for all $x \in M \cap \Omega$, then for $x \in M \cap \Omega_1$, $\Omega_1 \subset \Omega$, where Ω_1 is a domain bounded by the surface $V_0(x) = h$ (h is some constant). When passing through the domain with respect to the surface M the derivative $V_{*1}^{\cdot}(x)$ changes sign. For convenience we shall further assume that the domains Ω, Ω_1 are coincident.

We can note two simple special cases for choosing the required λ_j.

Case 1. There is at least one number $k_* \in \{m+1, \ldots, n\}$, for which $\partial\Phi_{k_*}(x)/\partial x_{k_*} \not\equiv 0$ and $\partial\Phi_{k_*}(x)/\partial x_k \equiv 0$ for $k \in \{m+1, \ldots, n\}$, $k \neq k_*$.

In this case it is sufficient to assume $\lambda_{k_*} = 1$, $\lambda_j = 0$, $j \in \{1, \ldots, n\}$, $j \neq k_*$ for inequality (6) to hold.

Case 2. For all $k, q = m+1, \ldots, n$; $i = 1, \ldots, m$ and for at least one $i_* \in \{1, \ldots, m\}$ $\partial\Phi_k/\partial x_q \equiv 0$, $\partial\Phi_{i_*}/\partial x_k \equiv 0$, $\partial\Phi_{i_*}/\partial x_i \not\equiv 0$.

To satisfy inequalities (6) it is here sufficient to assume $\lambda_{i_*} = 1$, $\lambda_j = 0$, $j = \{1, \ldots, n\}$, $j \neq i_*$. (In some cases the arbitrariness of the choice of the function $\Phi_{0i}(x)$ can also be used to reconstruct Lyapunov's function).

The inequalities (6) signify that the manifold $M_{11} \subset \Omega$, in which $V_{*1}^{\cdot} = 0$, does not agree with the manifold M and the cross-section $M_1 = M \cap M_{11}$ has dimensions which are smaller by at least one than those of M.

Therefore, if we can indicate the number $\mu_1 > 0$ for which the functions $V_0(x) + \mu_1 V_{*1}(x)$, $(-V_0^{\cdot}(x) - \mu_1 V_{*1}^{\cdot}(x))$ are positive on $\Omega\setminus\{0\}$ and $\Omega\setminus M_1$ respectively, then the sum $V_0(x) + \mu_1 V_1(x)$, $V_1(x) = V_{*1}(x)$ will represent the new Lyapunov function for which the manifold M_1 degenerates into a point or has dimensions which are less than those of M.

In the first case (for (2)) $p = 1$, and in the second, using the scheme described, we can construct the following function V_2 (for (2)), the constant μ_2, the manifold M_2 etc. up to V_p, μ_p for p, for which $M_p = \{0\}$.

If we cannot immediately indicate the number μ_1 then, bearing in mind the continuity of the function $V_{*1}(x)$ in Ω and the equation $V_{*1}(0) = 0$, we can choose the numbers δ and μ_{11} $(0 < \delta < 1,\ 0 < \mu_{11} \leqslant 1)$ and the natural number N_1, such that the compact $U_0 = \{x \in \Omega : \|x\| \leqslant \delta\}$ lies in the domain Ω and at the same time the following inequalities hold:

$$(-V_{*1}^{2N_1-1}(x)) \leqslant \frac{1}{2} V_0(x),\ x \in U_0;\ (-\mu_{11}) V_{*1}^{2N_1-1}(x) \leqslant \frac{1}{2} \min_{x \in U} V_0(x)$$

(U is the closure of the domain $\Omega \setminus U_0$).

It is obvious that the inequalities (6) hold for the time-derivative of the function $V_{*1}^{2N_1-1}(x)$ by virtue of (1), as for $V_{*1}^{\cdot}(x)$, only if $V_{*1}(x) \not\equiv 0$ on M. (This additional condition is assumed to hold henceforth).

The inequalities (6) will also hold in the domain G, $M \subset G$, enclosed between the surfaces $S_i(x) = \pm\varepsilon$ $(i = 1, \ldots, m)$ for fairly small $\varepsilon > 0$.

Further, the natural number N_2 is found, for which the following inequality will hold in the closure of the domain $U_0 \setminus G$:

$$\frac{d}{dt} V_{*1}^{2N_2-1}(x) \leqslant \frac{1}{2} |V_0^{\cdot}(x)|$$

(the derivative on the left is calculated by virtue of (1)) and the number $\mu_{12} \in (0, 1]$ is found, for which

$$\mu_{12} \frac{d}{dt} V_{*1}^{2N_2-1}(x) \leqslant \frac{1}{2} \min_{x \in U \setminus G} |V_0^{\cdot}(x)|$$

Hence it follows that, assuming $\mu_1 = \min\{\mu_{11}, \mu_{12}\}$, $N = \max\{N_1, N_2\}$ and $V_1 = V_{*1}^{2N-1}$, we will obtain a new Lyapunov function $V_0 + \mu_1 V_1$ with the above-mentioned properties.

Example 1. The set of equations /2/

$$x_1^{\cdot} = -x_1 - 3x_2^2, \; x_2^{\cdot} = -x_1 x_2 - x_2^3$$

admits of the Lyapunov function $V_0 = \frac{1}{2}(x_1^2 + x_2^2)$, and the manifold M is the parabola $x_1 = x_2^2$.

Since here

$$f_1^\circ = 2x_2^2, \; \Phi_1(x_1, x_2) = -2x_1 x_2^2 + \Phi_{01}(x_2)$$
$$f_2^\circ = -2x_2^3, \; \Phi_2(x_1, x_2) = \frac{1}{2} x_2^4$$

(which corresponds to Case 1), then, assuming $\lambda_1 = 0, \lambda_2 = 1$ in (5), we will obtain $V_1 = {}^1/_2 x_2^4$, and for any $\mu_1 > 0$ the function V is positive definite, and

$$V^{\cdot} = -x_1^2 + 2x_1x_2^2(1 - x_2^2) - x_2^4(1 + 2x_2^2) \quad (\mu_1 = 1)$$

is negative definite in all the phase plane.

Example 2. The set of equations /3/

$$x_1^{\cdot} = x_2, \; x_2^{\cdot} = -ax_2 + x_3, \; x_3^{\cdot} = -\psi(x_1) - \varphi(x_2)$$
$$a > 0, \; \psi(x_1) \in C^1(R^1), \; \varphi(x_2) \in C^1(R^1), \; \psi(0) = \varphi(0) = 0$$

when the following conditions hold:

$$\text{a)} \;\; \psi(x_1)x_1 > 0, \; x_1 \neq 0$$
$$\text{b)} \;\; a\varphi(x_2)/x_2 - \psi'(x_1) > 0, \; x_2 \neq 0; \; \psi'(x_1) = d\psi/\partial x_1$$

admits of the positive definite Lyapunov function

$$V_0 = a\int_0^{x_1} \psi(\xi)\,d\xi + \psi(x_1)x_2 + \int_0^{x_2} \varphi(\xi)\,d\xi + \frac{1}{2}x_3^2$$

for which the derivative $V_0^{\cdot} = [\psi'(x_1) - a\varphi(x_2)/x_2]\,x_2^2$ vanishes in the plane $x_2 = 0$ by virtue of condition b).
We have

$$f_1^0 = 0, \; f_2^0 = x_3, \; f_3^0 = -\psi(x_1)$$
$$\Phi_1 = \Phi_{01}(x_2, x_3), \quad \Phi_2 = -x_2x_3 + \Phi_{02}(x_3), \quad \Phi_3 = \psi(x_1)x_3 + \Phi_{03}(x_2)$$

Assuming $\Phi_{01}(x_2, x_3) = {}^1/_2 x_2^2, \; \Phi_{02} = \Phi_{03} = 0$, we will obtain

$$\sum_{i=1}^{3} f_i \frac{\partial \Phi_1}{\partial x_i} = -ax_2^2 + x_2x_3$$
$$\sum_{i=1}^{3} f_i \frac{\partial \Phi_2}{\partial x_i} = -x_3^2 + [\psi(x_1) + \varphi(x_2) + ax_3]\,x_2$$
$$\sum_{i=1}^{3} f_i \frac{\partial \Phi_3}{\partial x_i} = -\psi^2(x_1) + \psi(x_1)\varphi(x_2) + f'(x_1)x_2x_3$$

Assuming, then that $\lambda_1 = \lambda_2 = \lambda_3 = 1$ (this choice of coefficients here frees us from determining V_2), we will have

$$V_1 = {}^1/_2 x_2^2 + \psi(x_1)x_3 - x_2x_3, \quad V_1^{\cdot} = -ax_2^2 - \psi^2(x_1) - x_3^2 +$$
$$x_2x_3[\psi'(x_1) + a + 1] + [\psi(x_1) + \varphi(x_2)]\,x_2 - \psi(x_1)\varphi(x_2)$$

Since $V_1^{\cdot}$ when $x_2 = 0$ does not change sign, the domain Ω_1 can be chosen arbitrarily, and the number μ_1 is then defined for the new Lyapunov function $V = V_0 + \mu_1 V_1$.

Example 3. In /4/ the problem of stabilization in a field of the central force of the circular motion of a particle, controlled by the reaction u, is solved. The equation of the pertrubed motion of a material particle

$$x_1^{\cdot} = x_2, \; x_2^{\cdot} = v + bu, \; x_3^{\cdot} = (1 + x_1/r)\,u \tag{7}$$
$$u = -\frac{bx_2 + w}{\beta}, \quad v = -\frac{\mu}{(r + x_1)^2} + \frac{(\sqrt{\mu r} + x_3)^2}{(r + x_1)^3}$$
$$w = \frac{\sqrt{\mu r} + x_3}{r(r + x_1)} + \frac{1}{r^3}(\lambda r^2 x_3 - \sqrt{\mu r})(x_1 + r); \quad \mu, r, b, \beta, \lambda = \text{const} > 0$$

lends itself to a positive definite Lyapunov function /4/, the time-derivative of which, by virtue of (7), vanishes on the manifold M, defined by the equation $bx_2 + w = 0$. Solving this equation for x_2, we obtain

$$f_1^0(x_1, x_3) = -w/b, \; f_2^0(x_1, x_3) = v, \; f_3^0(x_1, x_3) \equiv 0$$

Assuming $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$, in the neighbourhood of the point $x = 0$ we have

$$V_{*1} = \Phi_{01}(x_3) + \frac{\lambda r}{2b}x_3 + \frac{1}{b}\left(\frac{1}{r^2} + \lambda\right)x_1x_3 - \frac{\sqrt{\mu r}}{br^3}x_1^2 + o(x^2)$$

To eliminate the linear term, we shall use the arbitrariness in choosing the function Φ_{01}, setting

$$\Phi_{01}(x_3) = -\frac{\lambda r}{2b}x_3$$

Then $V_1 = V_{*1}$.
Assuming (to simplify the notation) that $b = 1/r, \lambda = 4/r^2$, we will find the quantity

$$V_0^{\cdot} + \mu_1 V_1^{\cdot} = -2\beta u^2 + \mu_1 \;\; \{[\ln(1 + x_1/r) + x_1^2/r^2 + 4x_1/r] \times (1 + x_1/ru) - x_2(\beta ru + x_2)\}$$

which is negative on $M \setminus \{0\}$ and vanishes on M_1, determined by the equations

$$x_2 \equiv 0, \quad x_3 = \frac{\sqrt{\mu r}\,(2r - x_1)\,x_1}{5r^2 + 8rx_1 + 4x_1^2}$$

Further, determining V_2, we can show that the derivative $V_2^{\cdot}$ is negative definite on M_1 i.e. $M_2 = \{0\}$.

For system (1) of order two we can show that if the function V_0 is known and in the domain Ω the equation $S(x_1, x_2) = 0$, which describes the manifold M, is uniquely solvable for x_2 (or x_1), and the phase trajectories intersect M without touching, then the following inequality occurs in $\Omega \setminus \{0\}$:

$$f_1 \frac{\partial S}{\partial x_1} + f_2 \frac{\partial S}{\partial x_2} \neq 0$$

such that the unknown function V can be constructed always.

In fact, after transferring in (1) to polar coordinates using the equations $x_1 = r\cos\theta$, $S(x_1, x_2) = r\sin\theta$, $\theta \in [0, 2\pi]$ (or $x_1 = r\sin\theta$, $S(x_1, x_2) = r\cos\theta$) the manifold M coincides with the straight line $\theta = 0$, $\theta = \pi$, on which the derivative $\theta^{\cdot}$ is determined by the set of equations $\theta^{\cdot} = \vartheta(\theta, r)$, $r^{\cdot} = R(\theta, r)$, which corresponds to (1), unlike zero for $r \neq 0$.

Without loss of generality, we can assume that

$$\theta^{\cdot} = \vartheta(0, r) > 0, \quad \theta^{\cdot} = \vartheta(\pi, r) \gtrless 0, \quad r > 0$$

If the inequality $\vartheta(\pi, r) > 0$ occurs (the point $x_1 = 0$, $x_2 = 0$ is a stable focus), we will determine the periodic function, odd with respect to θ, of the period 2π,

$$V_1 = \begin{cases} -r^{\lambda} \sin 2k\theta, & |\theta|, |\theta - \pi| \leqslant \frac{\pi}{4k} \\ r^{\lambda} \sin \frac{2k}{2k-1}\left(\theta - \frac{\pi}{2}\right), & -\pi\left(1 - \frac{1}{4k}\right) \leqslant \theta \leqslant -\frac{\pi}{4k} \end{cases}$$

and if the inequality $\vartheta(\pi, r) < 0$ occurs, we will determine

$$V_1 = \begin{cases} -r^{\lambda} \sin k\theta, & |\theta|, |\theta - \pi| \leqslant \frac{\pi}{2k} \\ -r^{\lambda}, & \frac{\pi}{2k} \leqslant \theta \leqslant \pi\left(1 - \frac{1}{2k}\right) \end{cases}$$

(k is an odd natural number, λ is the lowest degree of the expansion of the function $V_0(r\cos\theta, r\sin\theta)$ with respect to the powers r).

The function V_1, determined in this way, is continuous in Ω together with the partial derivatives, and its time derivative, by virtue of (1), has the form

$$V_1^{\cdot} = \begin{cases} -2k\vartheta_1 \cos 2k\theta - R_1 \sin 2k\theta, & |\theta|, |\theta - \pi| \leqslant \frac{\pi}{4k} \\ \frac{2k}{2k-1}\vartheta_1 \cos \frac{2k}{2k-1}\left(\theta - \frac{\pi}{2}\right) + R_1 \sin \frac{2k}{2k-1}\left(\theta - \frac{\pi}{2}\right) \\ \frac{\pi}{4k} \leqslant \theta \leqslant \frac{4k-1}{4k} \end{cases}$$

when $\vartheta(\pi, r) \leqslant 0$;

$$V_1^{\cdot} = \begin{cases} -k\vartheta_1 \cos k\theta - R_1 \sin k\theta, & |\theta|, |\theta - \pi| \leqslant \frac{\pi}{2k} \\ -R_1, & \frac{\pi}{2k} \leqslant \theta \leqslant \pi \frac{2k-1}{2k} \end{cases}$$

when $\vartheta(\pi, r) \geqslant 0$. Here $\vartheta_1 = r^{\lambda}\vartheta(\theta, r)$, $R_1 = \lambda r^{\lambda-1} R(\theta, r)$.

Since the value $k = k_1$ exists, for which the derivative $V_1^{\cdot}$ will be negative definite on M, then for $k = k_2 > k_1$ the number ε, $\pi(4k) > \varepsilon > 0$, will be found for which $V_1^{\cdot}$ will be negative definite in the sectors $|\theta| < \varepsilon$, $|\theta - \pi| < \varepsilon$.

It further remains to determine the value $\mu_1 > 0$, for which the functions $V = V_0 + \mu_1 V_1$, $(-V^{\cdot})$ will be positive definite in Ω.

Note that this reconstruction of Lyapunov's function can be used to estimate the time of arrival of the phase point in a specified domain /5/.

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THE APPROXIMATE SYNTHESIS OF PERTURBED NON-VIBRATING SYSTEMS WITH ONE DEGREE OF FREEDOM*

L.D. AKULENKO

The pattern of the synthesis of a control which is optimal in speed of resp response for non-vibrating systems of a quite general form with one degree of freedom is discussed. The results of an analysis of such systems by the maximum principle /1/ are used; these results are based on constructing the switching curve of a relay control /2/. The picture of an approximate synthesis in the neighbourhood of a quiescent point (the origin of coordinates) obtained for controlled vibrating systems by asymptotic methods is complemented by the results obtained /3/.

1. Statement of the problem of synthesis that is optimal as regards speed of response for perturbed non-vibrating systems. *1.1. The initial control problem.* Consider the following perturbed controlled dynamic system with one degree of freedom:

$$x^{\cdot} = y, \quad y^{\cdot} = f(x, y, u) + \varepsilon F(x, y, u) \quad (x, y) \in G \subseteq R_2; \quad x(0) = x^0, \; y(0) = y^0 \tag{1.1}$$

Here x, y are the system's coordinate and its velocity, i.e. the generalized phase variables, R_2 is the phase plane, a dot means differentiation with respect to time $t \in [0, T]$ $(T < \infty)$; u is a scalar control piecewise-smooth function such that $|u(t)| \leqslant 1$; $\varepsilon \in [0, \varepsilon_0]$ is a small numerical parameter $(0 < \varepsilon_0 \ll 1)$, and f, F are smooth functions of x, y and u in the domain under consideration (the perturbation function F may be continuously dependent on ε). The additional properties (smoothness, growth, etc.) of the functions f and F, and of the domain G are discussed below. It should be noted that the constraints on the control u of the form $r^-(x, y, \varepsilon) \leqslant u \leqslant r^+(x, y, \varepsilon)$ are reduced to those discussed by the linear change

$$u = {}^1/_2 (r^+ + r^-) + {}^1/_2 (r^+ - r^-) v, \quad v \in [-1, 1]$$

(where v is the new control).

For the perturbed system (1.1) we formulate the problem of defining the law of the control that is optimal regarding speed of response in the form of the synthesis of $u(x, y, \varepsilon)$ which, for sufficiently small $\varepsilon > 0$, brings the phase point $(x, y) \in G$ to the origin of coordinates (the point $(0, 0) \in G$). It is assumed that the solution of the optimal synthesis for the unperturbed problem $(\varepsilon = 0)$ is known and is in the form of a control switching curve of a relay character /1, 2/.

Below we discuss the case of non-vibrating systems (non-oscillating objects, /2/), for which the unperturbed switching curves have the simplest form: the curve consists of two semitrajectories of the unperturbed system (1.1), going to the origin and corresponding to the constant extreme values $u = \pm 1$. In /2/ the sufficient conditions are given under which the synthesis of the control $u(x, y)$, optimal regarding speed of response in the whole of the plane R_2, or in a certain open domain $G \subset R_2$ which includes the neighbourhood of the origin, and has qualitatively the same form as that for the simplest dynamic system (1.1): $x^{\cdot\cdot} = u$, $|u| \leqslant 1$. Namely, "each optimal control has no more than one switching, and the switching line passes from the second to the fourth quadrant touching the x_2-axis $(x_2 = y)$ at the origin" (see /2/).

The sufficient conditions of this picture of the optimal synthesis are as follows (see /2/). It is assumed that the function f is continuously differentiable with respect to all arguments and satisfies the monotonicity condition with respect to u

$$f_u'(x, y, u) > 0, \quad (x, y) \in G, \quad |u| \leqslant 1 \tag{1.2}$$

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